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The Orbit Structure of Uniformly Positive Cyclic Systems

GARRETT BIRKHOFF

Department of Mathematics, Harvard University, Cambridge, Massachusetts 02138

AND

LEON KOTIN

Communications/Automatic Data Processing Laboratory, U.S. Army Electronics Command, Fort Monmouth, New Jersey 07703

1. INTRODUCTION

In this paper, we apply the concepts introduced in the immediately preceding paper [3] to obtain further insight into the orbit structure of third-order positive cyclic systems studied in [1, 2].

For this purpose, we first extend (in Section 2) to autonomous *families* of differential systems some old ideas of G. D. Birkhoff's theory of dynamical systems [5]. In particular, we extend to autonomous families the definition of *minimal* sets and observe that they always exist on compact manifolds. We next make (in Section 3) some general observations, which are immediate corollaries of our earlier work, about uniformly positive cyclic systems of arbitrary order.

We then consider autonomous families of third-order uniformly positive cyclic systems and examine the projections on the unit sphere S^2 of their *orbits* (i.e., phase space trajectories). We determine on S^2 the elliptic, parabolic, and hyperbolic points and the minimal sets of these families. We describe (in Theorem 9) a *periodic* solution which (projectively) bounds an "equatorial belt" containing *all* the oscillatory solutions. Finally, we show that the asymptotic qualitative behavior (as $t \to \pm\infty$) can be described quite well by reference to this belt and an orthogonal "polar cap" containing all the *positive* solutions.

2. SOME GENERAL CONCEPTS

We next recall some general definitions, applicable to any autonomous family of differential systems; cf. [6, 8, 9]. The terminology and notation will be that of [3].

DEFINITION. Given the autonomous family Φ as in [3], a set $S \subset \mathscr{R}$ is *Φ-closed* when any trajectory $\mathbf{x}(t)$ which originates in S always remains in S. The *Φ-closure* of S is the smallest Φ-closed set containing S.

In the notation of [3, (3)], S is Φ-closed if and only if $\mathbf{c} \in S$ implies $\Gamma(\mathbf{c}) \subset S$, or equivalently $T_{\mathbf{u}}(S, \tau) \subset S$ for any admissible $\mathbf{u} = \mathbf{u}(t)$ and $\tau > 0$. In the notation of [3, (2)], we also have evidently

$$\Gamma(S) = \bigcup_{\mathbf{c} \in S} \Gamma(\mathbf{c}) = \bigcup_{\mathbf{c} \in S} \left\{ \bigcup_{\tau \geqslant 0} K(\mathbf{c}, \tau) \right\}. \tag{1}$$

Closely related to the Φ-closure of a point $\mathbf{c}$ is its *omega set* [5, p. 197; 9, p. 50]:

$$\Omega(\mathbf{c}) = \bigcap_{\tau \geqslant 0} \overline{\bigcup_{t \geqslant \tau} K_\Phi(\mathbf{c}, t)}. \tag{2}$$

Some remarks about terminology: A Φ-closed set is termed strongly positive invariant in [8, p. 359; 9, p. 40] and invariant in [6, p. 20]; the sets $\Gamma(S)$ and $\Gamma(\mathbf{c})$ are called (right) funnels in [6, p. 14].

Immediate corollaries of the above definitions are the following.

COROLLARY 1. *Let S be any path-connected set of elliptic points of an autonomous family Φ. Then any Φ-closed set which includes one $\mathbf{x} \in S$ includes all of S.*

COROLLARY 2 [9, p. 42]. *Any union or intersection of Φ-closed (invariant) sets is itself Φ-closed (invariant).*

COROLLARY 3 [9, p. 41]. *The topological closure (in the usual topology) of any Φ-closed domain is Φ-closed.*

The following result generalizes a well-known property [7, p. 338] in the theory of dynamical systems.

COROLLARY 4. *The omega set $\Omega(\mathbf{c})$ of any point $\mathbf{c} \in \mathscr{R}$ is Φ-closed under the action of any autonomous family Φ of differential systems.*

COROLLARY 5. *For given $\mathbf{c}$, $\mathbf{x} \in \Omega(\mathbf{c})$ if and only if there exist a sequence of $t_n \uparrow \infty$ and associated control functions $\mathbf{u}_n(t)$ such that the solutions $\mathbf{x}_n(t)$ of the vector differential equation for the initial value $\mathbf{x}_n(0) = \mathbf{c}$ satisfy $\mathbf{x}_n(t_n) \to \mathbf{x}$.*

MINIMAL SETS. We define a ***minimal set*** as a nonvoid topologically closed and Φ-closed set in $\mathscr{R}$ which contains no proper nonvoid topologically closed and Φ-closed subset.

EXAMPLE 1. Consider the classical special case that $\Phi = \{\mathbf{X}(\mathbf{x})\}$ is a singleton. The concept of minimal set reduces to Birkhoff's notion of "central motion" [5, pp. 190–197]. Note that this case is always deficient and, except at critical points, hyperbolic.

Our next result is immediate, since the intersection of any two distinct minimal sets is topologically closed and Φ-closed, and contained in both. Since it cannot be smaller than either without being void, we conclude

COROLLARY 6. *Distinct minimal sets are disjoint.*

THEOREM 1. *Any nonvoid compact Φ-closed subset S for an autonomous family Φ contains a minimal set Σ.*

Proof. By Hausdorff's maximal principle, we can extend $\{S\}$ to a maximal nest of nonvoid compact Φ-closed subsets. The intersection of these will be a minimal (Φ-closed) subset Σ.

We define the *Φ-span* of a point $\mathbf{c} \in \mathscr{R}$ as the topological closure $\overline{\Gamma(\mathbf{c})}$ of its Φ-closure $\Gamma(\mathbf{c})$. By Corollary 3 above, this is the least Φ-closed and topologically closed set containing $\mathbf{c}$. Now suppose that Σ is minimal and $\mathbf{x} \in \Sigma$. Since $\overline{\Gamma(\mathbf{x})}$ is the least topologically and Φ-closed set containing $\mathbf{x}$, $\Sigma = \overline{\Gamma(\mathbf{x})}$. This proves half of

THEOREM 2. *A nonempty set Σ is a minimal set if and only if it is the Φ-span of any of its points* (i.e., $\Sigma = \overline{\Gamma(\mathbf{c})}$ *for any* $\mathbf{c} \in \Sigma$).

The converse is obvious.

The next result will be useful for us, though vacuous in the classical case (Φ a singleton) of an autonomous dynamical system, because in that case all points are deficient.

THEOREM 3. *Let Σ be a minimal set for a given autonomous family Φ. Then any nondeficient point* $\mathbf{q}$ *lying in Int Σ is attainable from any point* $\mathbf{p} \in \Sigma$.

Proof. First, suppose that the point $\mathbf{q} \in \operatorname{Int} \Sigma$ is *elliptic*. Then $\mathbf{q}$, being nondegenerate, is attainable from every $\mathbf{r}$ in some neighborhood of $\mathbf{q}$. Since $\Gamma(\mathbf{p})$ is dense in Σ, it contains some such $\mathbf{r}$; hence $\mathbf{q}$ is attainable from $\mathbf{p}$. On the other hand, suppose that the nondegenerate point $\mathbf{q} \in \operatorname{Int} \Sigma$ is *parabolic* or *hyperbolic*. From the autonomy of the family, $\mathbf{q}$ is attainable from every $\mathbf{r}$ in the curvilinear cone $C = \Gamma^-(\mathbf{q}) = \bigcup_{t \leq 0} K(\mathbf{q}, t)$ of t-decreasing trajectories through $\mathbf{q}$. (Thus C is diffeomorphic near $\mathbf{q}$ to an ordinary cone; instead of rays, it is bounded by negative trajectories.) Since $\mathbf{q}$ is nondegenerate, C is n-dimensional; so is $C \cap \Sigma$, since $\mathbf{q} \in \operatorname{Int} \Sigma$. This implies that $\Gamma(\mathbf{p})$ contains some $\mathbf{r} \in C$, by Theorem 2. Hence $\Gamma(\mathbf{p})$ contains $\mathbf{q}$.

But Σ is separable since $\mathscr{R}$ is finite-dimensional; hence if Σ is minimal and Φ is nowhere deficient in Int Σ, there is a trajectory which passes arbitrarily near every point of a countable dense subset of Int Σ. This proves the following result.

COROLLARY. *If Σ is minimal and Φ is nowhere degenerate in* Int Σ, *then Σ is the topological closure of a trajectory through any* $\mathbf{x} \in \operatorname{Int} \Sigma$.

DEFINITION. A set S in phase space $\mathscr{R}$ is *stable* for ("under the action of") an autonomous family Φ of vector fields [3, (1)] when it is the intersection of its Φ-closed open sets containing it; cf. [9, p. 76].

DEFINITION. For a given family Φ of vector fields $\mathbf{X}(\mathbf{x}, \mathbf{u})$, a *local* $\left\{\begin{smallmatrix} weak \\ strict \end{smallmatrix}\right\}$ *Liapunov function* at a point $\mathbf{c}$ is a differentiable function $v(\mathbf{x})$ such that

$$dv/dt = \Sigma(\partial v/\partial x_i)\, X_i(\mathbf{x}, \mathbf{u}) \qquad \begin{cases} \leqslant 0 \\ < 0 \end{cases}$$

for all admissible t, $\mathbf{u}$ and $\mathbf{x}$ with $0 < |\, \mathbf{x} - \mathbf{c} \,| < \epsilon$ for all sufficiently small ϵ.

The following results, though new, are almost immediate consequences of the relevant definitions; hence we omit the proofs.

THEOREM 4. *Let $v(\mathbf{x})$ be a local strict Liapunov function at $\mathbf{c}$ for Φ, and let S_k be the nonempty set of all points $\mathbf{y}$ such that $v(\mathbf{y}) \leqslant k$ for any constant $k \geqslant v(\mathbf{c})$. Then S_k is stable for Φ.*

For, we can set $U_\sigma(S_k) = \{\mathbf{x} \mid v(\mathbf{x}) < k + \sigma\}$.

THEOREM 5. *A family Φ of vector fields $\mathbf{X}(\mathbf{x}, \mathbf{u})$ admits a local strict Liapunov function near $\mathbf{x} = \mathbf{c}$ if and only if $\mathbf{c}$ is a hyperbolic point.*

3. POSITIVE CYCLIC SYSTEMS

We now apply the preceding observations to certain families of linear systems. G. D. Birkhoff observed long ago [4] that the qualitative behavior of low-order linear differential systems can be profitably studied *projectively* because this reduces the order by one. This observation applies equally to any *autonomous family* Φ of linear differential systems

$$d\mathbf{x}/dt = M(t)\mathbf{x}, \qquad M(t) \in \mathscr{M}, \tag{3}$$

$\mathscr{M}$ being any compact convex set of real $n \times n$ matrices M. We shall find it helpful to study linear systems also *spherically*, for essentially the same reason.

This reduction has a further advantage. Since projective and spherical n-space are *compact*, Theorem 1 ensures the existence of *minimal* sets for (3) when considered projectively *or* spherically.

Caution. The preceding reductions ordinarily change the type (elliptic, parabolic, hyperbolic) of points under the action of a given family $\mathscr{M}$ in (3), although both reductions give the same type because $\mathscr{V}(\mathbf{x})$ is unchanged relative to any given local coordinate system. However, we have the following result.

THEOREM 6. *If $\mathbf{x}_0$ is elliptic under $\mathscr{M}$ in* (3), *then it remains elliptic when $\mathscr{R}$ is considered projectively or spherically*; *if $\mathbf{x}_0$ is parabolic, it either stays parabolic or becomes elliptic.*

Proof. Ellipticity of $\mathbf{x}_0$ means that solutions through $\mathbf{x}_0$ can go in any direction; ellipticity or parabolicity means that $\mathbf{x}(t) \equiv \mathbf{x}_0$ is a solution. Both conditions are preserved under any differentiable mapping, even many-one.

EXAMPLE 2. Consider the special case of (3) where $\mathscr{M}$ is now the family $\mathscr{P}$ of nonnegative cyclic matrices $P(t) \equiv (p_{i,j})$ satisfying

$$0 \leqslant p_{i,i+1} \leqslant 1, \qquad p_{i,j} \equiv 0 \qquad \text{when} \quad j \neq i+1, \tag{4}$$

all subscripts being taken modulo n.

Since $\mathscr{P}$ contains the zero matrix, every point must be parabolic or elliptic. On the other hand, since every $P \in \mathscr{P}$ is nonnegative cyclic, the signs of the dx_i/dt compatible with a given x_i are all prescribed. Hence no point can be elliptic, and all points are parabolic.

But when considered spherically, we have the following result.

COROLLARY. *For the autonomous family* (4) *acting on spherical* $(n-1)$*-space, points in the interior of the positive and negative orthants are all elliptic. When n is even, this is also true of vectors whose components alternate in sign. All other points are parabolic. Moreover the above is also true projectively.*

We leave the proof to the reader.

Remark. As Professor Kakutani has observed (oral communication), the preceding reduction of first-order linear systems has the following interesting generalization. Let E by any *equivalence relation* on an n-dimensional phase-space $\mathscr{R}$ which "fibers" it into differentiable manifolds. Call a vector field $\mathbf{X}(\mathbf{x})$ compatible with E when $\mathbf{x}(0)\, E\mathbf{y}(0)$, $d\mathbf{x}/dt = \mathbf{X}(\mathbf{x})$ and $d\mathbf{y}/dt = \mathbf{X}(\mathbf{y})$ imply $\mathbf{x}(t)\, E\mathbf{y}(t)$—i.e., when any two solutions which are initially equivalent always stay equivalent (mod E). Then $d\mathbf{x}/dt = \mathbf{X}(\mathbf{x})$ also defines an autonomous system on the quotient space $\mathscr{R}/E$.

We remark that Kakutani's observation also holds more generally for autonomous families, but we will consider below only the cases that $\mathscr{R}$ is n-space with $\mathbf{0}$ deleted and the fibers are: (i) rays (sets $\{k\mathbf{x}\}$ with fixed $\mathbf{x} \neq \mathbf{0}$, taken for all $k > 0$) or (ii) projective points (sets $\{k\mathbf{x}\}$ with fixed $\mathbf{x} \neq \mathbf{0}$, and taken for all $k \neq 0$). In the first ("spherical") case, the equivalence $\mathbf{x}\tilde{E}\mathbf{y}$ will mean $\mathbf{x} = k\mathbf{y}$ for some $k > 0$, and in the second ("projective") case, $\mathbf{x}E\mathbf{y}$ will mean $\mathbf{x} = k\mathbf{y}$ for some $k \neq 0$. These fiberings are compatible with all linear (homogeneous) vector fields $\mathbf{X}(\mathbf{x}) = M\mathbf{x}$.

Uniformly positive cyclic systems. We now apply the preceding concepts to nth-order *uniformly* positive cyclic systems (cf. Example 2) of the form

$$dx_i/dt = p_i(t)x_{i+1}, \qquad 0 < m \leqslant p_i \leqslant Nm, \ i = 1, 2,..., n. \tag{5}$$

Note that, up to a scale of time, only N matters in what follows since the transformation $t \to t/m$ replaces m by 1 in (5).

In vector notation, (5) becomes

$$d\mathbf{x}/dt = P(t)\mathbf{x}, \tag{6}$$

where the coefficient matrix $P(t) = (p_{i,j})$ is nonnegative cyclic.

We recall the following result from [1, Theorems 1, 1′], in which *projectively unique* means unique to within a constant factor and *positive* solution means that all components are positive for all $t \in (-\infty, +\infty)$.

THEOREM A. *For any admissible* $\mathbf{p}(t) \equiv (p_1, p_2, ..., p_n)$, *there exist a projectively unique positive solution* $\mathbf{f}(t, \mathbf{p})$ *to* (5) *and a projectively unique positive solution* $\mathbf{g}(t, \mathbf{p})$ *to the adjoint differential equation*

$$d\mathbf{x}/dt = -P^T(t)\mathbf{x}, \tag{7}$$

where $P^T(t)$ *is the transpose of* $P(t)$.

For the constant-coefficient case $N = 1$, it is easy to see that spherically, when n is odd, there are exactly two fixed points, $\boldsymbol{\beta} \equiv (1, 1,..., 1)^T$ and $-\boldsymbol{\beta}$ (corresponding to the projectively unique positive and negative solutions $\pm e^{mt}\boldsymbol{\beta}$). When n is even, there are two additional fixed points, $\pm(1, -1,..., 1, -1)^T$. We now consider N as a parameter which increases continuously from $N = 1$ to $N = \infty$. The set $\tilde{\mathscr{E}}$ of all *elliptic or parabolic* points of $\mathscr{R}/E$ will expand continuously from the isolated fixed points of the case $N = 1$ (which are parabolic in that case) until they fill the entire positive hyperoctant and (for n even) the *alternating* hyperoctant whose elements have components which alternate in sign: $(+, -,..., +, -)$. This is a consequence of the following obvious fact, which also underlies the proof of the Corollary to Theorem 6.

THEOREM 7. *Let $\mathscr{M}$ be any convex compact set of real $n \times n$ matrices, and consider the action on $\mathscr{R}/E$* (resp. $\mathscr{R}/\tilde{E}$) *of the autonomous family* (3) *defined by the vector fields $\mathbf{X}(\mathbf{x}) \equiv M\mathbf{x}$ for some $M \in \mathscr{M}$. Then the set $\bar{\mathscr{E}}$ of points of $\mathscr{R}/E$* (resp. $\mathscr{R}/\tilde{E}$) *which are elliptic or parabolic under $\Phi(\mathscr{M})$ consists of the real eigenvectors of $\mathscr{M}$.*

COROLLARY. *As $\mathscr{M}$ increases, so does the set $\bar{\mathscr{E}}$ of elliptic or parabolic points under $\Phi(\mathscr{M})$.*

4. THIRD-ORDER UNIFORMLY POSITIVE CYCLIC SYSTEMS

In the rest of this paper, we shall consider projectively and spherically (i.e., in $\mathscr{R}/E$ and $\mathscr{R}/\tilde{E}$) the autonomous family $\Phi_{\mathscr{P}} = \{\mathbf{X}(\mathbf{x}, P): P \in \mathscr{P}\}$ of *third-order* uniformly positive cyclic systems defined, for some fixed $m > 0$ and $N > 1$, by $\mathbf{x}' = P(t)\mathbf{x}$, or

$$dx_i/dt = p_i(t)x_{i+1}\,, \qquad 0 < m \leqslant p_i(t) \leqslant mN, \qquad N > 1, \qquad i = 1, 2, 3, \quad (8)$$

where the subscripts are taken modulo 3. The index set $\mathscr{P}$ is the set of cyclic matrices P whose nonzero element in row i is p_i. As usual, we shall think of $\mathscr{R}/\tilde{E}$ as the unit sphere S^2. The projective plane $\mathscr{R}/E$ will be considered either as S^2 with antipodal points $\mathbf{x}$ and $-\mathbf{x}$ identified, or as a plane $x_1 = 1$ or $x_1 + x_2 + x_3 = 3$ augmented by a line at infinity.

The set of elliptic points. We next determine the set of elliptic points for the system Φ of (8), and then use this knowledge to compute its minimal sets.

THEOREM 8. *In* (8) *considered projectively* (*in* $\mathscr{R}/E$), *the set $\mathscr{E}$ of elliptic points is a curvilinear hexagon in the positive orthant, whose boundary $\partial\mathscr{E}$ consists of the six conics with $x_1 = 1$ and*

$$\begin{aligned} x_2x_3 &= N, & x_2 &= (1/N)x_3^2, & x_3 &= Nx_2^2, \\ x_2x_3 &= 1/N, & x_2 &= Nx_3^2, & x_3 &= (1/N)x_2^2. \end{aligned}$$

(Figure 1 shows these curves bounding $\mathscr{E}$, for $N = 1.5$, in the plane $x_1 = 1$. Figure 2 shows $\partial\mathscr{E}$ for $N = 1.5$, 3, 6, 12 in the plane $x_1 + x_2 + x_3 = 3$.)

Proof. As in Theorem 7, $\mathbf{x}$ is elliptic or parabolic precisely when $\mathbf{x}$ is a real eigenvector of some constant cyclic matrix $P \in \mathscr{P}$. Since this is impossible (for n odd) unless all x_i have the same sign, it suffices to consider the positive octant. Moreover, since the positive eigenvector of P varies continuously with P, the boundary $\partial\mathscr{E}$ of $\mathscr{E}$ consists of those $\mathbf{x}$ in the positive octant such that $P\mathbf{x} = \lambda\mathbf{x}$ for some P in $\partial\mathscr{P}$, the *boundary* of $\mathscr{P}$. But this is the rectilinear

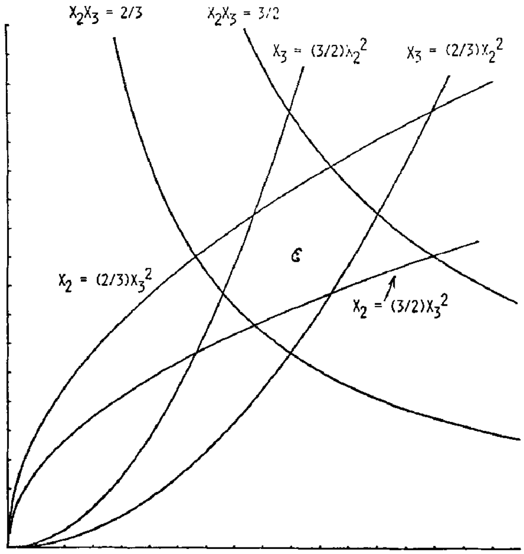

FIG. 1. The conic sections bounding $\mathscr{E}$, for $N = 1.5$, in the plane $x_1 = 1$.

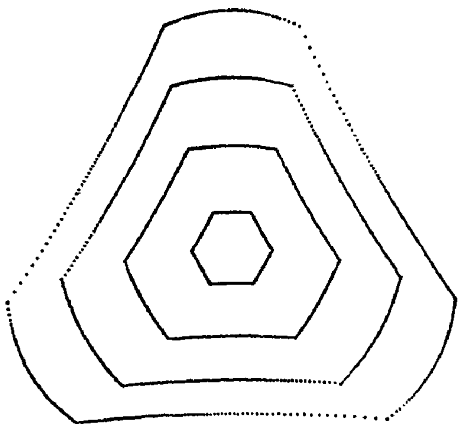

FIG. 2. $\partial\mathscr{E}$, for $N = 1.5, 3, 6, 12$, in the plane $x_1 + x_2 + x_3 = 1$.

hexagon $\mathscr{H}$ which corresponds in $\mathscr{R}/E$ to the cube $m \leqslant p_i \leqslant Nm$ in (p_1, p_2, p_3)-space. More precisely, the *cone* C of vectors $\mathbf{p}$ spanning $\mathscr{P}$ is subtended by this cube, so that ∂C is projectively the convex hexagon with the consecutive vertices $\mathbf{p}^j$, namely,

$$(1, 1, N), (1, N, N), (1, N, 1), (N, N, 1), (N, 1, 1), (N, 1, N). \tag{9}$$

Since boundaries are preserved under homeomorphisms, $\partial\mathscr{E}$ consists of the positive eigenvectors of $P(j, \sigma) \equiv (1 - \sigma)P^j + \sigma P^{j+1}$, $0 \leqslant \sigma \leqslant 1$, where the P^j are the cyclic matrices corresponding to the vertices $\mathbf{p}^j$ of $\mathscr{H}$. An elementary computation shows that these are the conics of the theorem.

In more detail, letting $\lambda = [\sigma N^2 + N - \sigma N]^{1/3}$ and $\lambda' = [N^2 - \sigma N^2 + \sigma N]^{1/3}$, the eigenvectors are

$$\mathbf{x}_{1\sigma} = \begin{bmatrix} 1 \\ \lambda \\ N/\lambda \end{bmatrix}, \qquad \mathbf{x}_{2\sigma} = \begin{bmatrix} 1 \\ \lambda' \\ \lambda'^2/N \end{bmatrix}, \qquad \mathbf{x}_{3\sigma} = \begin{bmatrix} \lambda \\ N/\lambda \\ 1 \end{bmatrix}$$

$$\mathbf{x}_{4\sigma} = \begin{bmatrix} \lambda' \\ \lambda'^2/N \\ 1 \end{bmatrix}, \qquad \mathbf{x}_{5\sigma} = \begin{bmatrix} N/\lambda \\ 1 \\ \lambda \end{bmatrix}, \qquad \mathbf{x}_{6\sigma} = \begin{bmatrix} \lambda'^2/N \\ 1 \\ \lambda' \end{bmatrix}.$$

Minimal sets. We next show how to determine ***numerically*** the ***minimal set*** Σ for the system (8) considered in the projective plane $\mathscr{R}/E$. By [1, Theorem 5; Theorem 6, Corollary 1] the elliptic point $\boldsymbol{\beta} \equiv (1, 1, 1)$ is projectively ***attainable*** from any point. Hence, by Theorem 2, the Φ-span $\overline{\Gamma(\boldsymbol{\beta})} = \Sigma$ of $\boldsymbol{\beta}$ is the ***unique minimal set*** for (8), considered projectively. Considered spherically (in $\mathscr{R}/\tilde{E}$), however, we have two minimal sets: Σ and $-\Sigma$.

To compute Σ numerically, we first project the positive octant onto the triangle

$$\Delta\colon x_1 + x_2 + x_3 = 3, \qquad \mathbf{x} > \mathbf{0} \tag{10}$$

At each hyperbolic point $\mathbf{x} \in \Delta$, the set $\mathscr{V}(\mathbf{x})$ is bounded by extreme directions. Recalling from [1] that the solutions of (8) outside the curvilinear hexagon of elliptic points have a clockwise sense as viewed from the positive octant, we next select for each hyperbolic $\mathbf{x} \in \Delta$ that one of the extreme vectors $\mathbf{p}^l = \mathbf{p}^l(\mathbf{x})$ where $l = l(\mathbf{x})$) in (9) which maximizes the outward direction on Δ of the trajectory at $\mathbf{x}$. We then form the cyclic system

$$dx_i/dt = p_i^l(\mathbf{x})x_{i+1}, \qquad i = 1, 2, 3 \tag{11}$$

with the components of these $\mathbf{p}^l(\mathbf{x})$ as coefficients. Although the system is nonlinear, the coefficient vector $\mathbf{p}^l(\mathbf{x})$ is constant in each of six subsets of the positive octant in $\mathbf{x}$-space. Integrating (11) numerically starting from $\partial\Delta$, computations show that the trajectory will spiral about $\partial\Sigma$ and approach it. (In particular, this shows that Σ is *stable*.)

This is shown in Fig. 3 for the cases $N = 1.5$, 3, 6, and 12 with the initial value $(\frac{3}{2}, \frac{3}{2}, 0)$ as a convenient choice.

This limiting curve is, by continuity, the orbit in $\mathscr{R}/E$ of a periodic solution of (8) for suitable $\mathbf{p}(t)$. Namely, we have:

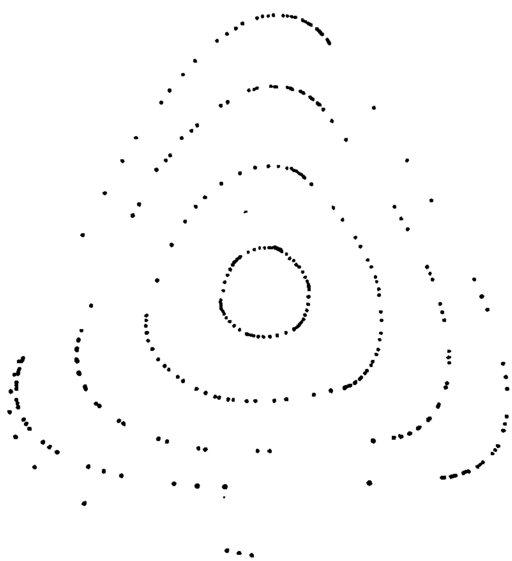

FIG. 3. Some points on $\partial\Sigma$, for $N = 1.5$, 3, 6, 12, in the triangle Δ. The "tail" corresponds to the integral curve of (11) and approaches $\partial\Sigma$ for $N = 12$.

THEOREM 9. *The boundary of the (unique) minimal set of* (8) *is a projectively periodic solution of* (8) *with piecewise constant* $\mathbf{p}(t)$, *unique up to a translation in* t.

Proof. From the computation of Σ, there is a piecewise constant vector $\mathbf{p}^l(\mathbf{x})$ and a solution of (8) whose trajectory approaches $\partial\Sigma$ as a limit cycle; i.e., $\partial\Sigma = \Omega(\mathbf{x})$ for any $\mathbf{x}$ on this trajectory. Then, from a well-known result on dynamical systems [7, p. 338], $\partial\Sigma$ is Φ^*-closed for the singleton family Φ^* consisting of the vector differential equation corresponding to $\mathbf{p}^l(\mathbf{x})$; i.e., any solution of this equation initially on $\partial\Sigma$ will remain there, and will be a (periodic) positive solution. Its uniqueness is easily proved by contradiction.

From this and the fact that any Φ-closed domain must have points in the positive or negative octant, it follows that Σ (in the positive octant) and $-\Sigma$ (in the negative octant) are the only minimal sets of the family (8) considered spherically.

Note that Theorem 9 also implies that for the autonomous family (8), any point in $\partial\Sigma$ is attainable from any other point in $\partial\Sigma$. Since no point of Int Σ is degenerate, with Theorem 3 this gives us the

COROLLARY. *If* $\mathbf{x} \in \partial\Sigma$, *then* $\Gamma(\mathbf{x}) = \Sigma$.

Companion to this result is the following.

THEOREM 10. *If* $\mathbf{x} \in \text{Int } \Sigma$, *then* $\Gamma(\mathbf{x}) = \text{Int } \Sigma$.

Proof. By Theorem 3, it remains to prove that $\partial\Sigma$ is not attainable from Int Σ. Suppose then that there exists $\mathbf{p}^0 = \mathbf{p}^0(t)$ with corresponding solution $\mathbf{x}(t, \mathbf{x}^0, \mathbf{p}^0)$ initially at $\mathbf{x}^0 \in \text{Int}\,\Sigma$, such that $\mathbf{x}(t_0, \mathbf{x}^0, \mathbf{p}^0) \in \partial\Sigma$ for some $t_0 > 0$. Consider $\mathbf{x}(t_0, \mathbf{x}, \mathbf{p}^0)$ for all $\mathbf{x} \in \text{Int}\,\Sigma$. The range of this topological mapping is open. But by hypothesis, $\mathbf{x}(t_0, \mathbf{x}^0, \mathbf{p}^0)$ belongs to the boundary of the open image-set. This contradiction establishes the result.

As a consequence, solutions which are on $\partial\Sigma$ for $t = 0$ cannot be in Int Σ for any $t < 0$. Since Σ is stable, the "critical periodic orbits" which constitute $\partial\Sigma$ behave like stable limit cycles of Ext Σ.

The preceding results give us the following characterization of Σ.

COROLLARY. *The minimal set Σ, considered spherically, coincides with the set Π of points in $\mathscr{R}/\tilde{E}$ lying on everywhere positive solutions of the autonomous family* (8).

Proof. From Theorem 9, $\partial\Sigma \subset \Pi$. Now for an arbitrary $\mathbf{x}^0 \in \text{Int}\,\Sigma$, let $\mathbf{x}(t, \mathbf{x}^0, \mathbf{p}^0)$ be the solution of (8) initially at $\mathbf{x}^0$, with $\mathbf{p} = \mathbf{p}^0$ any member of $\mathscr{P}$. From Theorem 10, $\mathbf{x}^1 = \mathbf{x}(1, \mathbf{x}^0, \mathbf{p}^0) \in \text{Int}\,\Sigma$, and $\mathbf{x}^0 = \mathbf{x}(t_1, \mathbf{x}^1, \mathbf{p}^1)$ for a suitable $\mathbf{p}^1$ and t_1. Repeating this gives us a (spherically periodic) positive solution through $\mathbf{x}^0$ and $\mathbf{x}^1$. Thus $\Sigma \subset \Pi$. Since $\Pi \subset \Sigma$ [1, Lemma 3], the conclusion follows.

5. THE EQUATORIAL BELT

We define a solution of (8) to be *oscillatory* when every component has arbitrarily large positive zeros. We recall from an earlier paper [1] that such solutions necessarily have arbitrarily large negative zeros and, also:

THEOREM B [1, Theorem 5]. *The values of the oscillatory solutions of* (8) *comprise a two-dimensional subspace for each t, namely, the plane* $\mathbf{g}^{\perp}(t, \mathbf{p})$ *orthogonal to the positive solution* $\mathbf{g}(t, \mathbf{p})$ *of the adjoint*; i.e., *a solution is oscillatory if and only if it lies in* $\mathbf{g}^{\perp}(t, \mathbf{p})$.

There is a curious duality between "positive" and "oscillatory" solutions of third-order positive cyclic systems, obtained by interchanging t and $-t$. This is illustrated by the following two dual results: (i) for fixed $\mathbf{p}(t)$, as $t \uparrow +\infty$ each *nonoscillatory* solution of the system (8) is asymptotic to a *positive* solution [1, Theorem 6, Corollary 1], and (ii) as $t \downarrow -\infty$ each *nonpositive* (or nonnegative) solution is asymptotic to an *oscillatory* solution [1, Lemma 2, Corollary 4].

It is suggestive to think of $\Sigma = \Sigma(\Phi)$ as a *polar cap* (the "arctic zone") about the "north pole" $\boldsymbol{\beta}/3$ on S^2 (see Fig. 4), and to introduce a dual

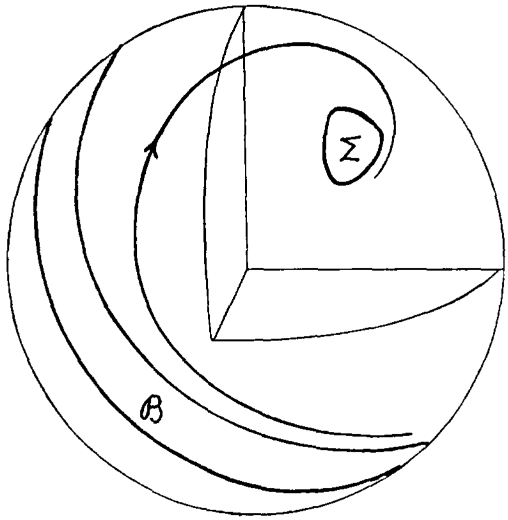

FIGURE 4

equatorial belt $\mathscr{B} = \mathscr{B}(\Phi)$, consisting of the trajectories of those solutions of (8) which are oscillatory on $(-\infty, \infty)$. As $t \to +\infty$, the polar cap Σ is *attractive*: any nonoscillatory solution either is ultimately in Σ or approaches $\partial\Sigma$ as a limit cycle (cf. [1, Theorem 6, Corollary 1]); oscillatory solutions remain in $\mathscr{B}$. For *decreasing* t, $\mathscr{B}$ becomes attractive: any nonpositive solution either is in $\mathscr{B}$ or approaches $\partial\mathscr{B}$ at $t \to -\infty$; positive solutions remain in Σ.

Dually, the equatorial belt can be viewed most simply as the ***minimal set*** for the family of ***negative*** cyclic systems obtained from Φ by time-reversal (the substitution $t \to -t$). By Theorem B, this is the set of values on the 2-sphere S^2 assumed by the oscillatory solutions of (8); moreover, as was shown in [1, Corollary 2 to Lemma 2; 2, Corollary to Theorem 6], this oscillatory behavior is stable as $t \downarrow -\infty$ for Φ, and hence as $t \uparrow \infty$ for the time-reversal of Φ.

When applied to Theorem 9, the preceding duality leads to the following result, obtainable by a construction used in that theorem (and which is likewise easy to implement computationally).

THEOREM 11. *There exist a* $P(t) \in \mathscr{P}$ *and a corresponding oscillatory periodic solution of* (8) *whose trajectory is projectively* $\partial\mathscr{B}$, *and spherically, one of the two edges of the equatorial belt* $\mathscr{B}(\Phi)$.

This leads us to the

COROLLARY. *For the family* (8), *if* $\mathbf{x} \in \partial\mathscr{B}$ *then* $\Gamma(\mathbf{x}) = \Omega(\mathbf{x}) = S^2 - \operatorname{Int} \mathscr{B}$.

Proof. From Theorem 11, $\mathbf{x} \in \Omega(\mathbf{x})$. Then from Corollary 4 in Section 2, $\Gamma(\mathbf{x}) \subset \Omega(\mathbf{x}) \subset \overline{\Gamma(\mathbf{x})}$. But $\Gamma(\mathbf{x}) = S^2 - \operatorname{Int} \mathscr{B}$, by an argument similar to that used in proving Theorem 10. This concludes the proof.

The adjoint. As in [1, Sect. 4], we now consider the *adjoint* of $\Phi = \Phi(\mathscr{P})$, namely, $-\Phi^T$, the autonomous family (7) associated with the negative transposes $-P^T$ of the matrices $P \in \mathscr{P}$. Since the undirected trajectories of the everywhere positive solutions for the adjoint $-\Phi^T$ are the same as those for Φ^T, the "polar caps" $\Sigma(-\Phi^T)$ and $\Sigma(\Phi^T)$, consisting of the values assumed by these solutions, are the same. From Theorem B, there follows:

THEOREM 12. *The equatorial belt $\mathscr{B}(\Phi)$ is the union of the great circles orthogonal to the vectors in the polar cap $\Sigma(\Phi^T)$ for the family of (positive cyclic) transposed systems* $\mathbf{x}' = P^T(t)\mathbf{x}$.

This polar cap $\Sigma(\Phi^T)$ can be obtained from $\Sigma(\Phi)$ by a simple transformation. The family Φ^T consists of systems of the form $dx_i/dt = p_i(t)x_{i-1}$, which is just the family Φ of cyclic systems (8) with the cyclic order reversed. Since this can also be obtained by interchanging x_i and x_{i+1} for any i in (8), $\Sigma(\Phi^T)$ can then be obtained from $\Sigma(\Phi)$ simply by a reflection through the plane $x_i = x_{i+1}$.

We now state without proof a number of easily proved consequences of the above and the results of [1, 2].

THEOREM 13. *For any family* (8), $\Gamma(\mathbf{x})$ *is given projectively by*: (i) S^2 *if* $\mathbf{x} \in \operatorname{Int} \mathscr{B}$, (ii) $S^2 - \operatorname{Int} \mathscr{B}$ *if* $\mathbf{x} \in \partial\mathscr{B}$, (iii) Σ *if* $\mathbf{x} \in \partial\Sigma$, (iv) $\operatorname{Int} \Sigma$ *if* $\mathbf{x} \in \operatorname{Int} \Sigma$. *Moreover, the Ω-sets are given projectively by*: (i) S^2 *if* $\mathbf{x} \in \operatorname{Int} \mathscr{B}$, (ii) $S^2 - \operatorname{Int} \mathscr{B}$ *if* $\mathbf{x} \in \partial\mathscr{B}$, (iii) Σ *if* $\mathbf{x} \in \Sigma$.

Again, denote by $\Gamma^-(\mathbf{x})$ the set of attainability of the point $\mathbf{x}$ for the uniformly *negative* cyclic system

$$\dot{\mathbf{x}} = -P(t)\mathbf{x}, \qquad P(t) \in \mathscr{P} \tag{12}$$

obtained from (8) by reversing time. Similarly, define the set of "alpha" limit points of (8) as the set $\Omega^-(\mathbf{x})$ of omega limit points of (12), as in [5, p. 197].

THEOREM 14. *For any family* (8), $\Gamma^-(\mathbf{x})$ *is given projectively by*: (i) $\operatorname{Int} \mathscr{B}$ *if* $\mathbf{x} \in \operatorname{Int} \mathscr{B}$, (ii) $\mathscr{B}$ *if* $\mathbf{x} \in \partial\mathscr{B}$, (iii) $S^2 - \operatorname{Int} \Sigma$ *if* $\mathbf{x} \in \partial\Sigma$, (iv) S^2 *if* $\mathbf{x} \in \operatorname{Int} \Sigma$. *Likewise*, $\Omega^-(\mathbf{x})$ *is*: (i) S^2 *if* $\mathbf{x} \in \operatorname{Int} \Sigma$, (ii) $S^2 - \operatorname{Int} \Sigma$ *if* $\mathbf{x} \in \partial\Sigma$, (iii) $\mathscr{B}$ *if* $\mathbf{x} \in \mathscr{B}$.

It would be interesting to compute the following three quantities as functions of N in (8); we have not done this:

(i) The periods of the projectively periodic solutions bounding Σ and B, whose existence is assured by Theorems 9 and 11.

(ii) The minimum and maximum times between successive zeros of components of oscillatory solutions.

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